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ON THREE SIMILAR FIGURES, WITH AN EXTENSION OF FEUERBACH'S THEOREM.

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X.

ON THREE SIMILAR FIGURES, WITH AN EXTENSION OF FEUERBACH'S THEOREM. By W. S. M'CAY, M.A., F.T.C.D.

[Read June 24, 1889.]

1. The fundamental properties of three directly similar figures in a plane are to be found in *Mathesis*, vol. ii. p. 73 (Tarry and Neuberg), and also in Casey's *Sequel to Euclid.* In a former Paper* I investigated some properties of three circles which occur in connexion with the theory of three similar figures, but only for the particular case where the figures were

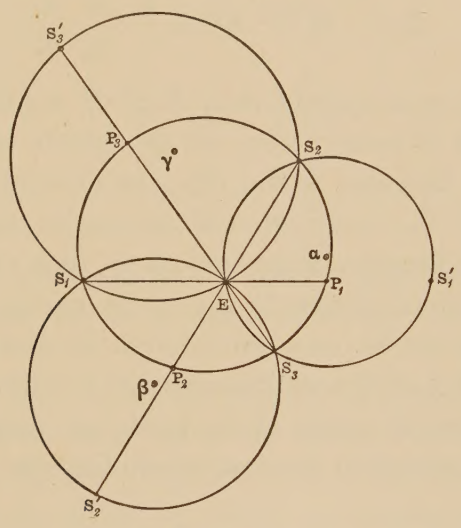

Fig. 1.

generated by the vertices of three similar triangles described on the sides of a fixed triangle. Many of the theorems proved apply with slight modification to a general system of three similar figures. If S_1, S_2, S_3 be the double

* *Transactions*, R. I. A., vol. xx., p. 453.

points, and E the director point, the three circles ES_2S_3, ES_3S_1, ES_1S_2 determine completely the whole systems F_1, F_2, F_3, and these circles are loci of collinear triads of corresponding points connecting through E. ES_1, ES_2, ES_3 cut the circle $S_1S_2S_3$ (circle of similitude) in the invariable points P_1, P_2, P_3 (fig. 1).

2. In every system of three similar figures (F_1, F_2, F_3) if A', B', C' be three corresponding points, A'', B'', C'' their inverses respectively with regard to the three circles ES_2S_3, ES_3S_1, ES_1S_2, then the triangles $A'B'C'$, $A''B''C''$ are inversely similar, E is their double point, and, if the diameter $A'A''$ of the circle ES_2S_3 cut it in UV, EU, EV, are the double lines.

This appears as follows:—If the species of $A'B'C'$ is given, the ratios EA': EB': EC' are given, for E is the mean centre of $A'B'C'$ for the fixed multiples $\frac{\sin \alpha_1}{a_1} : \frac{\sin \alpha_2}{a_2} : \frac{\sin \alpha_3}{a_3}$. If then EA': EB': $EC' = l : m : n$, we will have for the point C'

$$E_{13}C' : E_{23}C' : EC' = \frac{l}{a_1} : \frac{m}{a_2} : \frac{n}{a_3},$$

where E_{13}, E_{23} are the correspondents in F_3 of E, considered first as an F_1 point, and then as an F_2 point; they are the points of intersection of the circle ES_1S_2 with the tangents to the other two circles at E. Hence there are two points, C', C'', the intersection of two circles cutting ES_1S_2 orthogonally, and therefore inverses with respect to that circle, and of course A', A'' and B', B'' are inverses to the other circles. If we take three collinear points on the circles (on a line through E), and divide the diameters of the circles through these points internally and externally, all in the same ratio, we get two triads of points of the kind, and they are always seen to be inversely similar, except in the case where the sets of points coincide *on* the circles.

The director point E is similarly situated in the two similar triangles as being the mean centre of each set of vertices for the same multiples. The lines EU, EV cut the other two circles on the corresponding diameters $A'A''$, $B'B''$; hence these are rectangular double lines of the two triangles, for EU, EV are the bisectors of the angle $A'EA''$, since $A'A''$ are inverses to the circle ES_2S_3.

This result may be stated thus—"If we take the inverses of any three corresponding points A', B', C' with respect to the corresponding circles ES_2S_3, &c., the new points A'', B'', C'' will be corresponding points, vertices of a triangle inversely similar to $A'B'C'$. E is the double point, and the extremities of the diameters $A'A''$, &c. of the three circles lie on the double lines.

If Δ' be the area of the triangle whose vertices are three corresponding points A', B', C', it is not difficult to show that

$$\Delta' = \frac{\pi_1'}{r_1^2}\Delta\alpha\beta\gamma,$$

where π_1' is the power of A' with respect to the circle ES_1S_2 (radius r_1), and α, β, γ, the centres of the three circles. For the similar triangle $A''B''C''$ we will have

$$\Delta'' = \frac{\pi_1''}{r_1^2}\Delta\alpha\beta\gamma,$$

whence

$$\frac{1}{\Delta'} - \frac{1}{\Delta''} = \frac{1}{\Delta\alpha\beta\gamma},$$

$$\frac{\Delta'}{\Delta''} = \frac{\alpha A'}{\alpha A''}.$$

These equations are analogous to those which connect the areas of the pedal triangles of two inverse points with respect to the circumcircle of a triangle.

3. If we apply this to the triangle $P_1P_2P_3$, made by the invariable points, which is known to be inversely similar to the triangle made by three corresponding lines, we arrive at a new triangle $Q_1Q_2Q_3$, directly similar to the triangle made by three corresponding lines, and whose vertices are corresponding points. Parallels through the vertices of the triangle $Q_1Q_2Q_3$ form a triangle ABC; the sides of this triangle are corresponding lines, and their middle points are corresponding points. The perpendiculars at the middle points of the sides are therefore concurrent corresponding lines, and, accordingly, pass through P_1, P_2, P_3, and intersect on the circle $P_1P_2P_3$.

If α, β, γ be the centres of the circles ES_2S_3, &c., the centre of the circle ABC is at O, the intersection of αP_1, βP_2, γP_3, as in the Brocardian geometry.

This triangle ABC appears to be the nearest approach in a general system to the fundamental generating triangle in the Brocardian geometry. In that case, I pointed out that the sides of ABC were the polars of $A_1B_1C_1$ with regard to the respective circles, and it appears now in general that the sides of ABC are the polars of $P_1P_2P_3$ to the respective circles ES_2S_3, &c.

4. The lines joining A, B, C to S_1, S_2, S_3 meet in a point k on the circle of similitude, and the distances of k from the sides of ABC are as $a_1 : a_2 : a_3$. Now, the distances of E from the sides of $Q_1Q_2Q_3$ are as

$$\frac{1}{a_1} : \frac{1}{a_2} : \frac{1}{a_3};$$

hence k is the isogonal conjugate in the triangle ABC of what E is in the similar triangle $Q_1Q_2Q_3$. Parallels through k to the sides of ABC meet the perpendiculars at O in $P_1P_2P_3$. Ok is a diameter of the circle of similitude. This result as to the triangle ABC and the middle points of its sides is identical with one obtained by Dr. Kiehl,* by a very different method in investigating the position of the director point of three similar figures generated by perpendiculars to a triangle at the middle points of the sides, the three perpendiculars being in a constant ratio. We now see that the properties of this triangle, ABC, which exists for every three similar figures, follows simply from the inversion theorem of 2. Being given this triangle, and either of the points E or k, the whole system is determined.

5. The three corresponding lines αP_1Q_1, βP_2Q_2, γP_3Q_3 intersect on the circle of similitude. By inverting this circle into itself from the point E as origin, we obtain the following theorem for any triangle $P_1P_2P_3$, and a point E. Let P_1E, P_2E, P_3E cut the circle $P_1P_2P_3$ in $S_1S_2S_3$, and let E_1, E_2, E_3 be the reflexions of E with respect to the sides of $P_1P_2P_3$, W_1, W_2, W_3, the reflexions of S_1, S_2, S_3; then the three circles $EE_1S_1W_1$, $EE_2S_2W_2$,

* Programm-Abhandlung, Bromberg, 1888.

$EE_3S_3W_3$ intersect in a point O' on the circle $P_1P_2P_3$. It is not difficult to see that O' is the focus of a parabola inscribed in $P_1P_2P_3$, and whose directrix passes through E.

6. The triangle $\alpha\beta\gamma$ made by the centres of the three circles along with the point E constitute a triangle inversely similar to the modular triangle* of F_1, F_2, F_3; for S_1, S_2, S_3 are the reflexions of E with respect to the sides of $\alpha\beta\gamma$, respectively. (See fig. 1.) The distances from E to α, β, γ (the radii of the three circles) are as $a_1 : a_2 : a_3$, and the angles subtended by E at the sides are $\pi - \alpha_1$, $\pi - \alpha_2$, $\pi - \alpha_3$.

Hence, if we take E' the isogonal conjugate of E in the triangle $P_1P_2P_3$, and PQR the pedal triangle of E' in $P_1P_2P_3$, then E' has the same position in PQR that E has in the similar triangle $\alpha\beta\gamma$. No modular triangle can be constructed with an actual triad of three corresponding points, except the indeterminate infinite triangle whose vertices are the inverses of $\alpha\beta\gamma$ with respect to the three circles. The triangle $\alpha\beta\gamma$ becomes in the Brocardian geometry one similar to that made by the medians.

7. If any three corresponding circles be described round α, β, γ as centres, the orthogonal circles of all such sets of circles are coaxal and have E for a limiting point.

The powers of α, β, γ, with respect to such an orthogonal circle are as $a_1^2 : a_2^2 : a_3^2$, constant ratios; therefore the ratios of the perpendiculars from α, β, γ on the radical axis of the circle determined by the three points α, β, γ, and any circle orthogonal to the circles whose radii are θa_1, θa_2, θa_3 are given, and this radical axis is therefore fixed; it is the fixed external axis of similitude of three variable circles round $\alpha\beta\gamma$. Three particular circles round $\alpha\beta\gamma$ intersect at E, namely the circles ES_2S_3, &c., therefore E is a point circle of the orthogonal set, *i.e.* a limiting point; the other limiting point is the inverse of E to the circle $\alpha\beta\gamma$.

8. Let us consider the case of three similar figures when the director point is on the circle $\alpha\beta\gamma$. The double points S_1, S_2, S_3, being always reflexions of E with respect to the sides of $\alpha\beta\gamma$, will now be in one right

* M. Neuberg uses the expression "modular triangle" to denote a triangle and a point S, such that the three triangles made by S and each pair of vertices are directly similar to the triangles made by the double points, respectively, with pairs of corresponding points.

line parallel to the pedal line of E in the triangle $\alpha\beta\gamma$, and at twice the distance from E, for the double points are always the reflexions of E with respect to the sides of $\alpha\beta\gamma$. The line $S_1S_2S_3$ is the directrix of a parabola inscribed in $\alpha\beta\gamma$ whose focus is E. The circle of similitude becomes the line $S_1S_2S_3$ and the line at infinity; P_1, P_2, P_3 are at infinity in the directions S_1E, S_2E, S_3E, *i.e.* perpendicular to the sides of $\alpha\beta\gamma$. The triangles $\alpha\beta\gamma$, $Q_1Q_2Q_3$ coincide. The triangle ABC (3) has α, β, γ for the middle points of its sides.

The lines AS_1, BS_2, CS_3 are perpendicular to the line $S_1S_2S_3$, for $S_1S_2S_3$ is parallel to the pedal line of E in $\alpha\beta\gamma$, therefore perpendiculars from $\alpha\beta\gamma$ on the line meet in the isogonal conjugate at infinity of E in the triangle $\alpha\beta\gamma$; the point corresponding to E in the triangle ABC is the anticomplementary with respect to G, and its pedal line is, therefore, also parallel to $S_1S_2S_3$; that is, AS_1, BS_2, CS_3, are perpendiculars from ABC on $S_1S_2S_3$, and this line passes through the circumcentre O of ABC, for this is the orthocentre of $\alpha\beta\gamma$, a point on the directrix of the parabola before mentioned. The "adjoint points" S_1', S_2', S_3', become the reflexions of E with respect to the sides of ABC. The lines drawn from any point V on the line $S_1S_2S_3$ to $P_1P_2P_3$ at infinity (*i.e.* perpendicular to the sides of ABC) are corresponding concurrent lines, and determine similar rows of points on the sides of ABC.

10. These results may be stated in the following manner for a triangle:—

Let S_1, S_2, S_3 be the feet of perpendiculars from the vertices of a triangle ABC (fig. 2) on any line L drawn through the circumcentre O; α, β, γ the middle points of the sides; XYZ the pedal triangle of a variable point V on L. As V moves along L, X, Y, Z determine similar rows of points on the sides; the ratios of similitude $a_1 : a_2 : a_3$ being $\cos A' : \cos B' : \cos C'$, where A', B', C' are the angles at which the line cuts the sides: these are corresponding lines of three similar systems F_1, F_2, F_3, whose double points S_1, S_2, S_3 are collinear. VX, VY, VZ go through the invariable points at infinity.

If E be the director point, then α, β, γ, the middle points of the sides of ABC, are the centres of the three circles ES_2S_3, ES_3S_1, ES_1S_2. This of course

follows from 9, but I will prove it otherwise. Let V' be the inverse of V to the circumcircle, then we know that the pedal triangles XYZ, $X'Y'Z'$ are inversely similar; hence the pairs of points XX', YY', ZZ' must be inverses to the circles ES_2S_3, &c. (2); therefore these circles have their centres on the sides; but S_1, S_2 are equidistant from the middle point α of

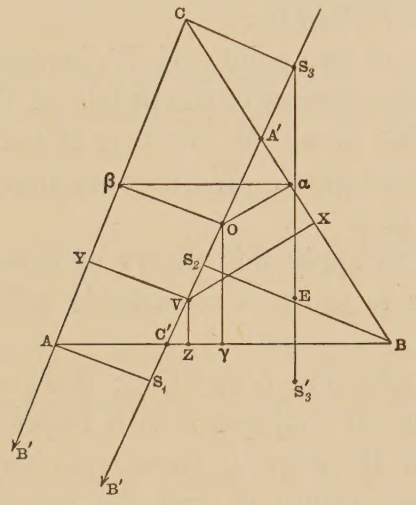

Fig. 2.

BC, since BS_2, CS_3 were drawn perpendicular to L; therefore α is the centre of the circle ES_2S_3, and similarly for the others. We see that the circle XYC (on VC as diameter) always passes through S_3, the focus of the parabola enveloped by XY, and similarly for the others. The circles $\alpha\beta$ CS_3, and $\alpha\beta\gamma$ (nine-point circle) are reflexions with respect to $\alpha\beta$; hence we see that E is the common reflexion of $S_1S_2S_3$ with respect to the corresponding sides of $\alpha\beta\gamma$, and that $S_1S_2S_3$ is the directrix of a parabola inscribed in $\alpha\beta\gamma$, whose focus is E.

The point E is the intersection of the pedal lines of the points U, U', where L cuts the circumcircle: the feet of those pedals lie on the three circles E, S_2, S_3, &c.—in fact the perpendiculars from any point V on L cut the three circles in two collinear triads of points connecting through E. E is the intersection of perpendiculars from S_1, S_2, S_3, on the sides of ABC. The radii of the three circles are $R \cos A'$, $R \cos B'$, $R \cos C'$.

When perpendiculars from the vertices of one triangle on the sides of another are concurrent, we know that the perpendiculars from the vertices of the other on the sides of the first are concurrent. We have a limiting case here for the flat triangle $S_1S_2S_3$, and we learn several interesting properties of the point of concurrence of the perpendiculars from the vertices of the flat triangle on the sides of the other. An extension to skew lines will be given further on.

The pedal triangles of two points, V, V', inverses with respect to the circumcircle, have the intersection of the pedals of V, V' for double point, and the pedals for double lines. M. Neuberg stated this result in a letter to me (April 28th) without proof. We now see that it is a particular case of the general theorem in 2.

11. I propose now to apply this theory of three similar figures with collinear double points to prove a remarkable extension of Feuerbach's theorem.

Let ABC be a triangle, and L any line: there exist on L two points, real or imaginary, isogonal conjugates with respect to the triangle (the intersections of L with its isogonal transformation). Suppose the line drawn through the circumcentre O, and the isogonal conjugates on it real.

The pedal triangles of FF' will have their vertices on a circle, the auxiliary circle of the conic inscribed in ABC whose foci are F, F'. This circle is the orthogonal circle of three corresponding circles described round $\alpha\beta\gamma$ as centres; for if XYZ, $X'Y'Z'$ be the pedal triangles of FF', we have $\alpha X : \beta Y : \gamma Z = \alpha X' : \beta Y' : \gamma Z' = a_1 : a_2 : a_3$; therefore the powers of $\alpha\beta\gamma$ with respect to the auxiliary circle of FF' are as $a_1^2 : a_2^2 : a_3^2$; therefore, by 7, E is a limiting point of the circle $\alpha\beta\gamma$ and the auxiliary circle, but E is on the circle $\alpha\beta\gamma$, therefore the circles *touch* at E. If the line L be drawn through O and the incentre or an excentre of ABC, we have Feuerbach's theorem. When O itself is a focus of a conic inscribed in ABC, then H, the orthocentre, is the other focus, and the nine-point coincides with the auxiliary circle.

12. This theorem may be proved directly without reference to similar figures.

Let FF' be the foci of a conic inscribed in the triangle ABC, the line FF' passing through the circumcentre O. The two pedal triangles XYZ, $X'Y'Z'$ of FF' have their vertices in the auxiliary circle of the ellipse. Let $\alpha\beta\gamma$ be the middle points of the sides of ABC, and let FF' cut the sides at angles A', B', C', we will have

$$\alpha X \,.\, \alpha X' = OF \,.\, OF' \cos^2 A',\ \beta Y \,.\, \beta Y' = OF \,.\, OF' \cos^2 B',\ \gamma Z \,.\, \gamma Z' = OF \,.\, OF' \cos^2 C'.$$

These are the powers of $\alpha\beta\gamma$ with respect to the auxiliary circle.

If we take $\alpha\beta\gamma$ for the triangle of reference, and apply to it the equation

$$abc\,\Sigma\, ayz = (\Sigma ax) \,.\, (ap_1x + bp_2y + cp_3z),$$

which represents a circle with respect to which the powers of the vertices are p_1, p_2, p_3, we obtain

$$\sin A \,.\, \cos^2 A' \,.\, x + \sin B \,.\, \cos^2 B' \,.\, y + \sin C \,.\, \cos^2 C' \,.\, z$$

as the radical axis of the auxiliary circle and the nine-point circle $\alpha\beta\gamma$ of the triangle ABC.

This line touches the circle $\alpha\beta\gamma(\Sigma\sqrt{\lambda \sin A} = 0)$ at the point

$$\frac{1}{\cos A'} : \frac{1}{\cos B'} : \frac{1}{\cos C'};$$

for, when a line cuts thesides of a triangle ($\alpha\beta\gamma$ or ABC) at angles A', B', C', we always have $\Sigma \sin A \,.\, \cos A' \equiv 0$, also $\Sigma \sin A \,.\, \sin A' \equiv 0$.

It appears that if the pedal line of any point on the circumcircle of a triangle cuts the sides at angles A', B', C', then the co-ordinates of the point on the circle referred to the triangle are

$$\frac{1}{\cos A'} : \frac{1}{\cos B'} : \frac{1}{\cos C'},$$

a result easily verified geometrically.

The co-ordinates of the point diametrically opposite on the circumcircle will be

$$\frac{1}{\sin A'} : \frac{1}{\sin B'} : \frac{1}{\sin C'},$$

with a pedal line perpendicular to the former. (The equation of the pedal line of the first point is $\Sigma x \sin A \cot A' = 0$.) The points F, F' may be imaginary; but the pedal circle is always real, and may be constructed for any line through the circumcentre. We know that the circle on the axis minor of an ellipse is the locus of feet of perpendiculars on tangents to the ellipse let fall from two imaginary points (*anti-foci*) on the axis minor.

The theorem is thus established; but without the aid of the theory of three similar figures we would not have full information as to the relations between the point E, the line L, and the triangle ABC.

The extension of Feuerbach's theorem may now be stated as follows:—

Any line, L, drawn through the circumcentre of a triangle ABC is an axis of a conic inscribed in the triangle; the corresponding auxiliary circle of the conic touches the nine-point circle of the triangle ABC, and the point of contact is the focus of a parabola inscribed in the triangle whose vertices are the middle points of the sides of ABC, L being the directrix of the parabola.

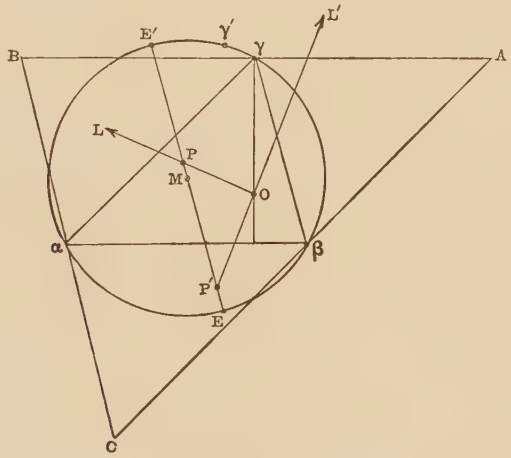

Fig. 3.

13. The following is a simple method of constructing the auxiliary circle of the inscribed conic whose axis passes through O, the circumcentre:—

The circle in fig. 3 is the nine-point circle of ABC, M its centre, O the circumcentre of ABC. Let OL be any diameter of the circumcircle. The required point of contact of the auxiliary circle is the focus of a parabola inscribed in $\alpha\beta\gamma$, and whose directrix is OL (O being the orthocentre of $\alpha\beta\gamma$). Draw $\gamma\gamma'$ parallel to OL, and draw $\gamma' E$ perpendicular to $\alpha\beta$ to cut the circle $\alpha\beta\gamma$ in E. It is easy to prove that the pedal line of E is parallel to $\gamma\gamma'$; therefore E is the focus. Since the auxiliary circle touches the nine-point circle at E, its centre is P, where EM cuts OL. The circle, whose centre is P and radius PE, cuts the sides in two triads of points, the perpendiculars at which intersect in two points on OP. If we draw the rectangular diameter OL, we arrive at E', the corresponding point on the nine-point circle for it. P', the corresponding centre of circle, is the intersection of $E'M$ and OL'. We see that on two rectangular diameters there are two centres of inscribed conics, and that two of the auxiliary circles touch the nine-point circle at E and E', the four points P, P', E, E' being on a diameter of that circle. In this figure both circles are major circles, for they cut the sides of ABC in real points.

The two pedal lines of diametrically opposite points on the circumcircle of a triangle are the actual asymptotes of the isogonal transformation (equilateral hyperbola) of the diameter with respect to the triangle, for they have the proper directions, and are easily seen to cut the sides at equal distances from the vertices.

14. The locus of foci (FF') of conics inscribed in the triangle ABC, when FF' passes through O, is a cubic through O; for F, F' are common harmonic conjugates with respect to the three pairs of points where the bisectors of the angles A, B, C cut the line FF'. The cubic is the locus of the points of contact of tangents from O to conics (equilateral hyperbolas) which pass through I, I_1, I_2, I_3, the incentre and excentres. Expressing that FF' passes through O, we obtain the locus of these points

$$\Sigma x(y^2 - z^2)\cos A = 0.$$

The locus of P, the middle point of FF', is another cubic, having O for a double point. As the line OL in its rotation round O passes an incentre

or excentre, the corresponding auxiliary circle changes character, from major to minor, or *vice versa.*

15. If we call the angles $MOP = \theta$, $OMP = \phi$, it is not difficult to see that $2\theta - \phi$ is constant. Hence the polar equation of the locus of P referred to O as origin, and OM as axis, is

$$\rho \sin(3\theta - \alpha) = c \sin(2\theta - \alpha).$$

We easily verify from this that rectangular lines from O cut the cubic in two points which connect through M. The tangents at O are rectangular, and there are three real asymptotes.

The trilinear equation of the locus of P referred to the triangle ABC is

$$\Sigma(y^2 \cos^2 C - z^2 \cos^2 B) \,.\, \Sigma x \sin A + 4 \sin A \,.\, \sin B \,.\, \sin C \,.\, \Sigma x(y^2 - z^2) \cos A = 0;$$

or, in another form,

$$\Sigma \frac{y \cos(A - B) - z \cos(C - A)}{y \cos B - z \cos C} \sin A = 0.$$

16. O is the actual centre of the inscribed conic when E, E' are the intersections of OM with the nine-point circle. The two auxiliary circles in this case belong to the conic whose centre is O. They are the circles described round O to touch the nine-point circle. The axes of this conic are parallel to the pedal lines of EE' in the triangle $\alpha\beta\gamma$.

17. The Brocard ellipse of the triangle ABC furnishes a case where the minor circle touches the nine-point circle. The line MS (S the centre of the ellipse is the middle point of $\Omega\Omega'$, the Brocard points) cuts the nine-point circle in two points, E and E'. The nearer of these (say E) to S is the point of contact of the minor circle. E is the Steiner's point of the triangle $\alpha\beta\gamma$, for we know that R is the focus of a parabola inscribed in ABC, whose directrix is parallel to OK; the complementary line OK in the triangle $\alpha\beta\gamma$ is parallel to OK. The other point is Tarry's point of $\alpha\beta\gamma$; it is the point of contact of an auxiliary circle of a conic inscribed in ABC, and of which an axis lies along the perpendicular at O to OK.

18. There are many simple cases of the general theorem.

One is: Let the radius CO of the circle ABC cut AB in C'; then CC' is an indefinitely slender conic of the kind considered. The circle on CC' as diameter touches the nine-point circle at the foot of the perpendicular from C on AB.

19. The following is another construction for the auxiliary circle of a conic inscribed in ABC, and of which an axis passes through the circumcentre. "Let the axis cut the sides at the points A', B', C', and S_1, S_2, S_3 be the feet of perpendiculars from ABC on it; take T_1, T_2, T_3, the harmonic conjugates of S_1, S_2, S_3, to $B'C'$, $C'A'$, $A'B'$, respectively; then the segments S_1T_1, S_2T_2, S_3T_3 have a common harmonic segment which is the diameter of the circle required. This construction follows from the property that any point on an axis of a conic has the same polar with respect to the conic and the corresponding auxiliary circle. The circle given by this construction must always be real, for we have seen how to construct it otherwise.

20. The conic inscribed in ABC, and of which the axis major passes through O, will be in three cases a parabola. This will happen when the tangent to the nine-point circle at the point E is parallel to the pedal line of E. These directions are parallel to the cuspidal tangents of Steiner's tri-cusped envelope of the pedal line, and are the directions of the asymptotes of the locus in 15. In the case of these lines the auxiliary circle is the tangent at the vertex of the parabola, and touches the nine-point circle.

21. There are six equilateral hyperbolas of the kind. For the centre of an equilateral hyperbola inscribed in ABC lies on the polar circle which cuts the cubic locus in six points.

22. Let the axis FF' of an inscribed conic pass through any fixed point P, and let LMN be the pedal triangle of P, with sides l, m, n. By a method similar to that of 12, taking LMN as triangle of reference, I

find that the equation of the radical axis of the auxiliary circle of the conic with the circle LMN is

$$l\cos^2 A'x + m\cos^2 B'y + n\cos^2 C'z = 0,$$

with the condition

$$\sin A\cos A' + \sin B\cos B' + \sin C\cos C' = 0.$$

Therefore the radical axis envelopes a conic circumscribed to LMN, whose equation is

$$\frac{a^2}{lx} + \frac{b^2}{my} + \frac{c^2}{nz} = 0.$$

This reduces to the case in 12, when L, M, N are the middle points of the sides of ABC.

The variable auxiliary circle will touch the fixed circle LMN four times at its point of contact of common tangents with the conic.

23. Let ABC be a triangle, and L any line (fig. 4). From any point

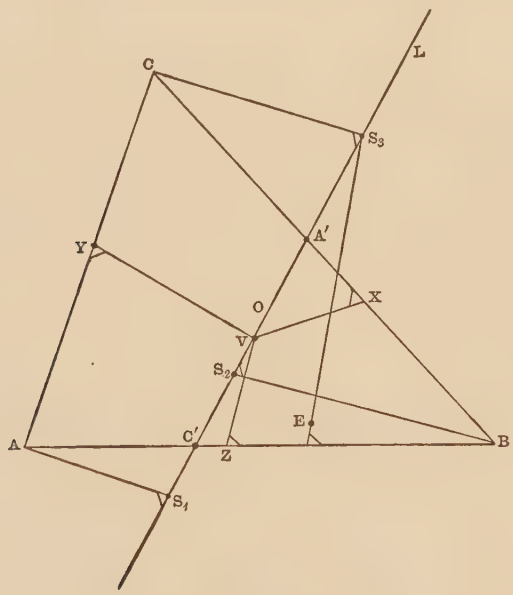

Fig. 4.

V of L draw VX, VY, VZ, making the same angle θ with the sides in

the same aspect. We may call the triangle XYZ the skew pedal of V at an angle θ. As V moves along L, θ remaining constant, X, Y, Z will make similar rows of F_1, F_2, F_3 on the sides. The angles of similitude are A, B, C, and the ratios of similitude are $\sin(A'+\theta) : \sin(B'+\theta) : \sin(C'+\theta)$, where A', B', C' are the angles at which L cuts the sides. The double points S_1, S_2, S_3 are got by drawing parallels from A, B, C, to cut L at an angle θ, in the opposite aspect to that in which VX, VY, VZ cut the sides. P_1, P_2, P_3 are at infinity in the directions VX, VY, VZ. Parallels from S_1, S_2, S_3, to VX, VY, VZ, intersect in E. E is also, by the general theory, the intersection of the skew pedal lines at an angle θ of the points where L cuts the circle ABC.

In the case where L passes through the circumcentre O of ABC, the centres of the three circles ES_2S_3, &c., will again have their centres on the sides of ABC, and at the vertices of the skew pedal at angle θ of O.

This may be proved directly, but it follows by the same reasoning as in 10, for the skew pedals at an angle θ of two points inverses to the circumcircle are similar, for it is well known that the triangle XYZ is similar to the perpendicular pedal triangle. The centres then of the three circles ES_2S_3, &c., are three corresponding points on the sides, and form a triangle similar to ABC; therefore they are the vertices of the skew pedal of O. E is, as before, the common reflexion of $S_1S_2S_3$ with respect to the sides.

In the general case where L does not pass through O, the triangle $\alpha\beta\gamma$ (centres of three circles ES_2S_3, &c.) has its sides equal and parallel to those of the skew pedal of O. For we will have the general case if the triangle ABC be moved parallel to itself along the lines AS_1, BS_2, CS_3 into any position $A''B''C''$, the points S_1, S_2, S_3, E remaining fixed. The skew pedal of the circumcentre of $A''B''C''$ moves equal and parallel to the fixed triangle $\alpha\beta\gamma$.

This might have have been derived from the case of perpendicular pedals by drawing any three parallel lines from S_1, S_2, S_3 (*i.e.* to a point at infinity on the circle of similitude), and placing on them a triangle similar to ABC.

In the case of perpendicular pedals we can find a point V on L, and a triangle $A''B''C''$ formed by three corresponding lines, such that V is the

incentre of $A''B''C''$. Let VX, VY, VZ cut the parallel triangle in $X''Y''Z''$. If we express that V is the incentre of $A''B''C''$, we have

$$r = VX + \theta \cos A',$$
$$r = VY + \theta \cos B',$$
$$r = VZ + \theta \cos C',$$

and VX, VY, VZ are x, y, z, the co-ordinates of V in ABC, whence

$$\Sigma x(\cos B' - \cos C') = 0.$$

This line is satisfied by the co-ordinates of the incentre of ABC, and by the point $\cos A' : \cos B' : \cos C'$, the point at infinity in a direction perpendicular to L (12). We have then V determined as the foot of the perpendicular from I, the incentre, on L. If round this point V we describe a circle equal to the incircle of ABC, the triangle circumscribed to this circle and parallel to ABC has its sides for three corresponding lines, and the points of contact of the incircle for corresponding points, a result obvious on inspection by making a figure. This is a case of the circles noticed by Dr. Casey, *Mathesis* vii. p. 14, when the two sets of points coincide. The auxiliary circle (11) is another case, the two triads being distinct.

24. The important property that E is the Mean Centre of all triads of corresponding points for the multiples

$$\frac{\sin \alpha_1}{a_1} : \frac{\sin \alpha_2}{a_2} : \frac{\sin \alpha_3}{a_3}$$

is not proved anywhere that I know of. It is stated in a note to *Mathesis*, vol. ii. p. 76. The following is a simple proof:—

Let ABC, $A'B'C'$ be two triads of corresponding points, then AA', BB', CC' intersect at angles α_1, α_2, α_3, and are in the ratios $a_1 : a_2 : a_3$.

These two sets of points will have a common mean centre for multiples $l : m : n$. Let pp', qq' be the perpendiculars from AA', BB' on the line CC', we will have

$$lp + mq = lp' + mq',$$

or

$$l(p - p') = m(q' - q),$$

that is,

$$la_1 \sin \alpha_2 = ma_2 \sin \alpha_1.$$

Similarly for the other sides; whence

$$\frac{la_1}{\sin \alpha_1} = \frac{ma_2}{\sin \alpha_2} = \frac{na_3}{\sin \alpha_3}.$$

If ABC be fixed, and $A'B'C'$ variable, we see that the mean centre for the stated multiples is fixed. But this point for the triangles $P_1P_2P_3$ and $\alpha\beta\gamma$ is E.

Starting from this fundamental property of E, the principal properties of three similar figures could be deduced, using figure 1.

NOTE ADDED IN THE PRESS.

25. I give another proof of the theorem in 11, 12, which puts the matter in a different light.

Let the circumcircle of a triangle ABC and any inscribed conic remain fixed while the triangle varies; then the nine-point circle of the triangle always touches two fixed circles concentric with the conic.

This appears easily as follows: It is known that the polar circle of the triangle ABC cuts orthogonally the director circle of the conic; hence the power of H, the orthocentre of ABC with respect to the circumcircle, is twice the power of H with respect to the director circle: H therefore describes a circle coaxal with these two circles, and its centre is the reflexion of O, the circumcentre with respect to P, the centre of the conic; hence the nine-point centre, middle point of OH, describes a circle round P, and its radius, $\frac{1}{2}R$, being constant, the above theorem is proved. Now suppose that the conic had been chosen so as to be in a position of symmetry with the circle (*i. e.* either axis passing through the circumcentre); then as the triangle ABC varies, it will be isosceles in two positions, with the base touching the conic at the extremity of that axis which passes through O; therefore one of the envelope circles of the nine-point circle is the auxiliary circle on that axis; hence, for any position of the triangle, if the conic is

symmetrically placed with respect to the circle, one of its auxiliary circles touches the nine-point circle.

The invariant relation between the circle and conic (Salmon's *Conics*, p. 342) gives the relations

$$OP^2 = a^2 - b^2 + R^2 \pm 2Ra,$$

or

$$OP^2 = b^2 - a^2 + R^2 \pm 2Rb,$$

according as O is on the major or minor axis. These coincide with relation $D^2 = R^2 \pm 2Rr$ when $a = b$.

NOTE.—I have just seen in the *Educational Times* of July that Mr. J. Griffiths has shown (reprint *E. T.*, vol. viii., p. 35, 1867) from the interpretation of a sextic equation which represents two coincident cubics, that the condition the pedal circle of a point with respect to a triangle should touch the nine-point circle is, that the line joining the point to its isogonal conjugate should pass through the circumcentre.